Amplification and Attenuation of Sound by Burning Propellants[1]

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From the point of view of determining the acoustic stability of a solid propellant rocket motor, it is vital to know whether or not a sound wave is amplified or attenuated upon reflection from the burning surface. If amplification occurs, instability may result. This question has two parts to its answer. The first part concerns the burning response of the combustion layer itself to an acoustic pressure fluctuation, and the second part concerns the matching of an incident and reflected sound wave at the surface in order to determine whether the reflected wave has been amplified or attenuated. In previous theoretical treatments, it has been customary to ignore the simultaneous existence of thermal or entropy waves when performing this matching procedure. In this study, the authors carry out the detailed matching of the thermo-acoustic field at the combustion zone boundary and examine the results in the light of previous studies on the acoustic stability of solid propellant rocket motors.

The ability of a burning solid propellant to amplify sound frequently leads to rocket motors functioning as very high-level acoustic oscillators. The deleterious effects on motor performance are frequently of great importance and are widely recognized. Thus, considerable practical importance has been attached to the measurement of the ability of propellants to amplify pressure disturbances. Such experimental measurements are now becoming available, and direct comparison with theoretical predictions will soon become possible for the first time. Consequently, it seems timely to re-examine the theoretical developments and to begin to refine and adapt them to the enlightenment of experimental results.

The underlying cause of the phenomenon is conceptually simple and has been discussed at length elsewhere (1).[4] In the thin burning "boundary layer," a rather intricate balance pertains in the steady state between gasification rate and flame speed. If the solid should begin to gasify at too fast a rate, the flame front will be pushed back, the flow of heat from flame to solid will be diminished, and the gasification rate will be reduced. An analogous compensatory action will result if the gasification rate should fluctuate downward or if the flame speed should fluctuate. As long as the burning zone has time to equilibrate between disturbances, the results will be quasi-static. But when the surface is subjected to a succession of disturbances, coherent in phase and closely spaced in time—as it may be when enclosed in an acoustic cavity such as a rocket motor—what will be the result? In order to determine the response of the burning surface to a sound wave, one should solve the associated equations obeyed by time-dependent chemistry and fluid dynamics. The problem is represented schematically in Fig. 1. To the right of some plane $x = x_f$, the mechanical properties of the burned gases are described by the usual acoustic equation; thus, when the burning zone is probed by an incident acoustic wave, the effect of the burning propellant on the sound field exterior to the boundary layer is described in terms of the acoustic admittance of the plane $x = x_f$. In general, however, the location of this plane will not easily be determined experimentally, so that the net effect is usually described empirically in terms of a virtual specific admittance associated with the location of the solid surface. Of course, when the boundary layer is thin compared with the acoustic wavelength, it is not necessary to distinguish between these two locations so far as the acoustic field is concerned.

In part, the virtual specific admittance of the surface is determined by the gross fluid dynamic conditions there, and, in part, it is intimately related to the detailed characteristics of the boundary layer. For this reason, it will be useful to define a dimensionless entity, the "reduced virtual specific admittance" y related to the virtual specific admittance Y by the relationship

$$Y = -(\bar{v}_f/\bar{P})y$$

where $\bar{v}_f$ and $\bar{P}$ are the mean flow velocity and pressure in the asymptotic region. It will be the problem of the theorist to illuminate the detailed anatomy of this reduced admittance, whereas the experimentalist can give it numerical values without being diverted by questions of the rigor of the theoretical treatments.

In his attempt to determine the virtual reduced admittance that the experimentalist measures, the theorist must recognize immediately that the measurement equipment of the experimentalist may be located in the far zone or in the near zone.

In the far zone, the actual admittance Y_f' easily is related to the local mass flow rate of the propellant gas. If the gas pressure, density, velocity, and mass flow rate are denoted by P, ρ, v, and m, respectively, and the incremental quantities corresponding to these are defined by $P = \bar{P}(1 + \epsilon)$, $\rho = \bar{\rho}(1 + \sigma)$, $v = \bar{v} + \phi$, and $m = \bar{m}(1 + \mu)$, where the bars indicate time averages, then

$$-Y_f' \equiv \frac{\tilde{\phi}_f}{\tilde{\epsilon}_f\bar{P}} = \frac{\bar{v}_f}{\bar{P}}\left[\frac{\tilde{\mu}_f}{\tilde{\epsilon}_f} - \frac{\tilde{\sigma}_f}{\tilde{\epsilon}_f}\right] \qquad [1a]$$

where the complex amplitude of a function f has been denoted by $\tilde{f}$, i.e., $f = Re(\tilde{f}e^{i\omega t})$. To the right of $x = x_f$, the usual sound field isentropic relationship $\tilde{\sigma}/\tilde{\epsilon} = 1/\gamma$ pertains (γ is the specific heat ratio), and Eq. [1a] may be written as

$$-Y_f' = \frac{\bar{v}_f}{\bar{P}}\left[\frac{\tilde{\mu}_f}{\tilde{\epsilon}_f} - \frac{1}{\gamma}\right] \qquad [1b]$$

If Y_f' is known, a transformation remains to be performed in order to express the virtual admittance which the experimentalist measures. The problem will be somewhat more complicated when measurements are made in the near zone because of the relatively complex wave motion of the gas in the anisentropic region.

In the near zone, close to the burning surface just exterior to the gas phase combustion sublayer, it will be appropriate to

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use the admittance Y', defined by the relationship

$$Y' = -\frac{\tilde{\phi}_b}{\tilde{\epsilon}_b \bar{P}} = -\frac{\bar{v}_b}{\bar{P}}\left(\frac{\tilde{\mu}_b}{\tilde{\epsilon}_b} - \frac{\tilde{\sigma}_b}{\tilde{\epsilon}_b}\right) \qquad [1c]$$

As one should expect, the ratio $\tilde{\sigma}_b/\tilde{\epsilon}_b$ will be unity at zero frequency and will remain near unity at low frequencies.

Thus, the central problem of the analyst must be the solution of the several physical and chemical equations that describe the burning rate response of the solid propellant to the acoustic field. The task is not an easy one, complicated as it must be by the intricacies of the poorly understood chemical processes themselves. Useful progress has been made, however, at a rather impressive cost in approximations, and it now seems advisable to examine previous work in an attempt to remove or refine such approximations as may submit to a new assault. In this study, only the acoustic boundary sublayer will be considered.

Acoustic Boundary Sublayer

In previous work on solid propellant rocket motor combustion instability, when attention has been focussed on the response of the burning rate of the propellant to pressure disturbances, it has been focussed primarily on the determination of μ_b. Then, in discussing the effect to be expected due to an impinging sound wave, it has been customary to collapse the acoustic boundary sublayer entirely (1–4). This is common practice in acoustics, where it usually corresponds to the neglect of the very small thermal loss from the sound field into the boundary wall. In the present case, however, the potential gain due to the burning propellant turns out to be very small also, and the effect of this collapse on the calculation of the admittance deserves critical examination. The order of magnitude of the effect to be expected, as pointed out by Summerfield (5) and others, is presumably indicated by considering the extreme case of the low frequency limit. In that limit, one may assume an isothermal rather than an isentropic acoustic field, and the $1/\gamma$ in Eq. [1b] would then be replaced by unity. Since the quantity μ_f/ϵ_f in Eq. [1b] is frequently not far from unity, the collapse of the acoustic boundary sublayer is often an untenable approximation.

The first task, then, will be that of expressing the admittance $Y(x)$ at any arbitrary plane where the pressures and velocities might be measured in terms of the quantities μ_b and ϵ_b, which have been estimated previously (2). The second task will be that of relating this admittance to the virtual admittance, which may be directly observed experimentally.

For present purposes, the dynamics of the gas in the region exterior to the combustion sublayer will be described by the equations of energy, momentum, and mass transport in only one dimension (plus the equation of state of an ideal gas). Thus, the problem will be rather more messy than difficult, and only the major signposts will be displayed along the way to the final result of the analysis. For a more detailed discussion of the solution of the basic equations, the reader may consult, for example, Ref. 6. A convenient representation of these equations is:

Energy

$$-(\partial/\partial x)(\tfrac{1}{2}\rho v^3 + \rho v C_p T) + (\lambda \partial^2 T/\partial x^2) = (\partial/\partial t)(C_v \rho T + \tfrac{1}{2}\rho v^2)$$

Momentum

$$(1/\rho)(\partial P/\partial x) + (v\partial v/\partial x) + (\partial v/\partial t) = 0$$

Mass

$$(\partial/\partial x)(\rho v) = -(\partial \rho/\partial t)$$

State

$$\rho = P/[(C_p - C_v)T]$$

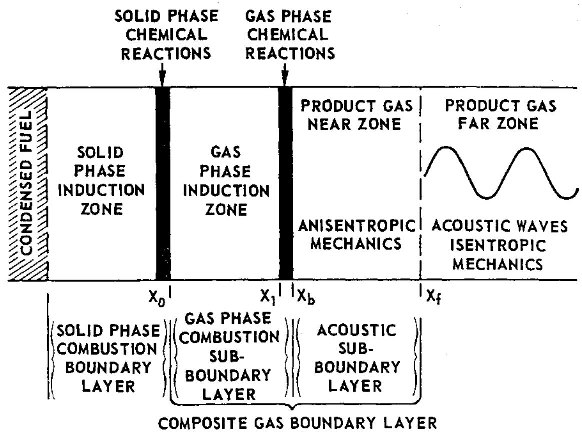

Fig. 1 Schematic diagram showing a cross section normal to the solid propellant surface

λ is the thermal conductivity, T is the temperature, and C_p, C_v are the heat capacities at constant pressure and constant volume, respectively. One should note here that the gas viscosity has been omitted, whereas thermal conductivity has been retained. Obviously, this will not be justifiable at frequencies so high that the visco-acoustic damping length becomes so short as to be comparable to the distance $x_f - x_b$, which is the distance scale of interest. But one will recall that in acoustic theory the thermal and viscous damping lengths are at least of the same order of magnitude, so that it might at first appear that, if thermal conductivity is considered, then viscosity should be retained also. However, in the present case, both of these damping lengths will be so long that their effect on the sound (isentropic) component of the gas motion will turn out to be negligible. The importance of retaining heat conduction arises because of its contribution to the damping of the nonisentropic component of the gas motion, in which the gas viscosity plays an insignificant role.

The next step will be to perturb about the steady state and discard all terms of second and higher order in the sound field quantities. In the region ($x \geqslant x_b$), one notes that the time average flow field quantities are independent of position, so that the four fundamental equations become

$$\left(\frac{\gamma - 1}{c^2}\right)\left(\frac{-\lambda\bar{T}}{\bar{\rho}}\frac{\partial^2\tilde{\psi}}{\partial x^2} + \frac{3}{2}\frac{\bar{v}^2\partial\tilde{\phi}}{\partial x} + \frac{\bar{v}^3}{2}\frac{\partial\tilde{\sigma}}{\partial x}\right) + \frac{\bar{v}\partial\tilde{\epsilon}}{\partial x} + \frac{\partial\tilde{\phi}}{\partial x} = \frac{-i\omega\tilde{\epsilon}}{\gamma} - \left(\frac{\gamma - 1}{c^2}\right)\left[\frac{i\omega\tilde{\sigma}v^2}{2} + i\omega\bar{v}\tilde{\phi}\right] \qquad [2a]$$

$$(c^2/\gamma)(\partial\tilde{\epsilon}/\partial x) + (v\partial\tilde{\phi}/\partial x) + i\omega\tilde{\phi} = 0 \qquad [2b]$$

$$(\bar{v}\partial\tilde{\sigma}/\partial x) + (\partial\tilde{\phi}/\partial x) + i\omega\tilde{\sigma} = 0 \qquad [2c]$$

and

$$\tilde{\psi} = \tilde{\epsilon} - \tilde{\sigma} \qquad [2d]$$

where

$$T \equiv \bar{T}(1 + \psi) \qquad c^2 \equiv \gamma\bar{P}/\bar{\rho} \qquad \gamma \equiv C_p/C_v$$

and where the time derivative operator has been replaced by $i\omega$ in the usual way, thereby confining attention to harmonic fields.

Eliminating ψ with the aid of Eq. [2d] and trying an x dependence of the form e^{sx} for the remaining unknown quantities in Eqs. [2a–2c], one immediately finds three equations that must be satisfied simultaneously. These may be written as

$$\{a_{j,1}\tilde{\epsilon} + a_{j,2}\tilde{\sigma} + a_{j,3}\tilde{\phi}\} = 0 \qquad j = 1, 2, 3 \qquad [3a]$$

where the a's are functions of s and are given as

$$a_{1,1} = \left(\frac{\gamma - 1}{\bar{\rho}c^2}\right)\lambda\bar{T}s^2 - \bar{v}s - \frac{i\omega}{\gamma}$$

$$a_{1,2} = -\left(\frac{\gamma - 1}{\bar{\rho}c^2}\right)\lambda\bar{T}s^2 - \frac{(\gamma - 1)}{2c^2}\bar{v}^3 s - \frac{1}{2}(\gamma - 1)\frac{i\omega(\bar{v})^2}{c^2} \quad [3b]$$

$$a_{1,3} = -\frac{3}{2}\left(\frac{\gamma - 1}{c^2}\right)(\bar{v})^2 s - s - \left(\frac{\gamma - 1}{c^2}\right)i\omega\bar{v}$$

$$a_{2,1} = c^2 s/\gamma \qquad a_{2,2} = 0 \qquad a_{2,3} = i\omega + \bar{v}s$$

$$a_{3,1} = 0 \qquad a_{3,2} = \bar{v}s + i\omega \qquad a_{3,3} = s$$

In order for a solution to exist, the determinant of the a's must vanish, and since the determinant is a quartic in s, this defines four "waves" out of which the complete solution is to be composed. Thus, one will have, in general,

$$\tilde{\epsilon} = \tilde{\epsilon}_1 e^{s_1 x} + \tilde{\epsilon}_2 e^{s_2 x} + \tilde{\epsilon}_3 e^{s_3 x} + \tilde{\epsilon}_4 e^{s_4 x}$$
$$\tilde{\phi} = \tilde{\phi}_1 e^{s_1 x} + \tilde{\phi}_2 e^{s_2 x} + \tilde{\phi}_3 e^{s_3 x} + \tilde{\phi}_4 e^{s_4 x} \quad [3c]$$

and analogous expressions for $\tilde{\sigma}$ and $\tilde{\psi}$. The four values s_j are the four roots of the determinant D which finally reduces to

$$0 = D = \frac{c^3}{\gamma}\left\{s^4\left[M^4(1 - \gamma M^2)\tau_e c\right] - s^3\left[\frac{2i\omega\gamma}{c}M^5\tau_e c\right] + s^2\gamma\frac{M^4\omega^2}{c^2}(\tau_e c) - \left(sM + \frac{i\omega}{c}\right)\left[(1 + M)s + \frac{i\omega}{c}\right] \times \left[(1 - M)s - \frac{i\omega}{c}\right]\right\} \quad [4a]$$

where

$$\tau_e \equiv [(\gamma - 1)\lambda T]/\rho\bar{v}^4 \qquad M \equiv \bar{v}/c$$

Although the analysis can be carried through formally without determining the four s_j's, it will be necessary to evaluate them for any numerical calculations. For that purpose, it will be convenient to order these roots according to their magnitude. The smallest s, say s_1, corresponds to the acoustic (isentropic) wave traveling to the right. One finds

$$s_1 = -[i\omega/c(1 + M)](1 + \Delta_a) \quad [4b]$$

where

$$\Delta_a = -i\omega\tau_e\left(\frac{\gamma - 1}{2}\right)\frac{M^4}{(1 + M)^2} + 0[(i\omega\tau_e M^4)^2]$$

The real part of s_1 corresponds to thermal damping of the acoustic wave and is very small for ordinary burning velocities, even at very high frequencies (e.g., for $M = 3 \times 10^{-3}$, $\tau_e = \frac{1}{5}$ sec, $c = 10^5$ cm/sec, $\gamma = 1.2$, one has real $s_1 \approx -10^{-5}$ cm^{-1} at 10^5 cps, corresponding to a damping length of 1000 m). The next larger root, s_2, corresponds to a sound wave incident from the right and is given by

$$s_2 = +[i\omega/c(1 - M)](1 - \Delta_a) \quad [4c]$$

From this point on, Δ_a will be ignored. The third root corresponds to thermal or entropy wave.[5] One finds that

$$s_3 = -(i\omega/Mc)(1 + \Delta) \quad [4d]$$

where

$$\Delta \approx (1/M^2\omega\tau_e)[(i/2) - M^2\omega\tau_e - (i/2)(1 + 4iM^2\omega\tau_e)^{1/2}] \quad [4e]$$

Eq. [4e] is correct to terms of order M^2 at low frequency and also at high frequency, provided that $M^2\omega\tau_e$ is not much greater than unity. Thus, the approximate expression Eq. [4e] will encompass the domain of Mach numbers and frequencies of interest here. (For a typical case, $M^2 = 10^{-5}$, $\tau_e = \frac{1}{5}$ sec, the quantity $M^2\omega\tau_e$ becomes unity at about 10^5 cps.)

The fourth and final root of the determinant is given to the same order of approximation as s_3 by the expression

$$s_4 = (i\omega/Mc)(1 + \Delta_t) \quad [4f]$$

where

$$\Delta_t \approx [1/M^2\omega\tau_e][(i/2) - M^2\omega\tau_e + (i/2)(1 + 4iM^2\omega\tau_e)^{1/2}] \quad [4g]$$

Thus, this fourth "wave" corresponds in the limit of zero flow or high frequency to that solution of the simple heat conduction equation which becomes infinite as $x \to +\infty$. (Note that, for typical values considered in the previous example, the real part of s_4 corresponds to a damping length of $\sim 10^{-3}$ cm.) This "wave" would correspond to a thermal source downstream, so for the present problem $\tilde{\sigma}_4$, $\tilde{\phi}_4$, $\tilde{\psi}_4$, $\tilde{\epsilon}_4$ vanish as a boundary condition.

Now return to a consideration of the solution to the conservation equations, as expressed by Eq. [3c], and to evaluation of the coefficients $\tilde{\epsilon}_j$, $\tilde{\phi}_j$, $\tilde{\psi}_j$, and $\tilde{\sigma}_j$. The first step will be that of selecting one of these quantities to play the role of independent variable and then expressing each of the others in terms of it. Here it will be convenient to select the pressure amplitude $\tilde{\epsilon}_2$ of the incident acoustic wave as independent variable. From the momentum and mass conservation equations, one easily obtains the relations

$$\tilde{\phi}_j = -\frac{c^2}{\gamma}\frac{s_j}{(i\omega + \bar{v}s_j)}\tilde{\epsilon}_j \quad [5a]$$

$$j = 1, 2, 3, 4$$

and

$$\tilde{\sigma}_j = \frac{c^2}{\gamma}\left(\frac{s_j}{(i\omega + \bar{v}s_j)}\right)^2\tilde{\epsilon}_j \quad [5b]$$

$$j = 1, 2, 3, 4$$

It will be necessary to determine two complex admittances, Y and Y', which are defined to be the ratio of acoustic (isentropic) velocity to acoustic pressure amplitudes and the ratio of total fluctuating velocity to total fluctuating pressure, respectively. Using Eq. [5a], one obtains

$$Y_b = -\frac{1}{\bar{P}}\left(\frac{\tilde{\phi}_1 + \tilde{\phi}_2}{\tilde{\epsilon}_1 + \tilde{\epsilon}_2}\right)\Bigg|_b = \frac{c^2}{\gamma\bar{P}}\frac{\left[\frac{s_1}{i\omega + \bar{v}s_1}\frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_2} + \frac{s_2}{i\omega + \bar{v}s_2}\right]}{\left(1 + \frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_2}\right)}\Bigg|_b \quad [6a]$$

[5] The terminology that seems most appropriate depends on the velocity and frequency regime of interest. If the mean velocity vanishes (or if the frequency becomes sufficiently high), this "wave" corresponds to the solution of the simple heat conduction equation

$$\lambda(\partial^2 T/\partial x^2) = (\partial/\partial t)(\rho C_v T)$$

which vanishes as $x \to +\infty$, and in this regime the term thermal wave or thermal wave front is apt. For usual flow velocities and frequencies not too much higher than 10^3 cps

$$s_3 \approx -(i\omega/Mc)[1 + \omega\tau_8 M^2 + \text{terms of higher order in } (i\omega\tau_8 M^2)]$$

Thus, in this domain, this third type of wave travels away from the surface at a speed equal to the mean flow speed, and with a damping length $\approx (c/\omega^2\tau_e M)$, which typically will be of the order of 5 cm at 1000 cps, whereas the wavelength is about 0.3 cm. In this regime, it is convenient to think of this component of the solution as an entropy wave.

Note that the distance $(x_1 - x_b)$ is negligibly small compared with the acoustic wavelengths of interest, so that Eq. [6a] expresses the virtual admittance referred to in the introduction. Further

$$Y_b' \equiv -\frac{\tilde{\phi}_b}{\tilde{\epsilon}_b\bar{P}} = -\frac{1}{\bar{P}}\frac{(\tilde{\phi}_1+\tilde{\phi}_2+\tilde{\phi}_3)}{(\tilde{\epsilon}_1+\tilde{\epsilon}_2+\tilde{\epsilon}_3)}\Bigg|_b = \frac{c^2}{\gamma\bar{P}}\frac{\left[\frac{s_1}{i\omega+\bar{v}s_1}\left(\frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_2}\right)+\frac{s_2}{i\omega+\bar{v}s_2}+\frac{s_3}{i\omega+\bar{v}s_3}\frac{\tilde{\epsilon}_3}{\tilde{\epsilon}_2}\right]}{\left[\frac{\tilde{\epsilon}_1}{\tilde{\epsilon}_2}+1+\frac{\tilde{\epsilon}_3}{\tilde{\epsilon}_2}\right]}\Bigg|_b \quad [6b]$$

Thus, it will be essential to evaluate the ratios $\tilde{\epsilon}_1/\tilde{\epsilon}_2$ and $\tilde{\epsilon}_3/\tilde{\epsilon}_2$ at $x = x_b$.

In order to complete the solution, one must apply the appropriate boundary conditions. Since theoretical considerations of the burning region will lead to specification of quantities at $x = x_b$, the boundary equations will, in general, relate the quantities $\tilde{\mu}_b$, $\tilde{\psi}_b$, and $G_b \equiv (\partial\tilde{\psi}/\partial x)_{x=xb}$. Thus, one must satisfy the following three equations:

$$\tilde{\mu}_b = \sum_{=1,2,3,4}\left(\tilde{\sigma}_j+\frac{\tilde{\phi}_j}{Mc}\right)e^{s_jx_b} = \sum_{j=1,2,3,4}\frac{c^2}{\gamma}\tilde{\epsilon}_j\left[\frac{(s_j)^2}{(i\omega+\bar{v}s_j)^2}-\frac{1}{Mc}\frac{s_j}{(i\omega+\bar{v}s_j)}\right] \quad [7a]$$

$$\tilde{\psi}_b = \sum_{j=1,2,3,4}\tilde{\psi}_be^{s_jx_b} = \sum_{j=1,2,3,4}\tilde{\epsilon}_j\left[1-\frac{c^2}{\gamma}\left(\frac{s_j}{i\omega+\bar{v}s_j}\right)^2\right] \quad [7b]$$

$$\tilde{G}_b = \sum_{j=1,2,3,4}s_j\tilde{\psi}_je^{s_jx_b} = \sum_{j=1,2,3\,4}\tilde{\epsilon}_js_j\left[1-\frac{c^2}{\gamma}\left(\frac{s_j}{i\omega+\bar{v}s_j}\right)^2\right] \quad [7c]$$

where x_b has been set equal to zero for convenience. Finally, one also requires $\tilde{\epsilon}_4 = 0$.

Thus, regarding $\tilde{\epsilon}_2$ as the independent, arbitrarily assigned quantity, Eqs. [7a–7c] constitute a set of three inhomogeneous linear equations in the three unknowns $\tilde{\epsilon}_1$, $\tilde{\epsilon}_3$, and $\tilde{\epsilon}_4$. The ancillary requirement that $\tilde{\epsilon}_4 = 0$ implies one relationship between $\tilde{\mu}_b$, $\tilde{\psi}_b$, and $\tilde{G}_b$, so that these three quantities cannot be specified entirely independently. Combustion zone theory will typically provide just the necessary two remaining independent boundary condition relationships. The first is a relationship between the fluctuating mass flow rate and the pressure, and the second is a relationship between the fluctuating temperature, temperature gradient, and pressure. This second relationship has, in linear perturbation theory, the general form

$$\tilde{\psi}_b = B_p\tilde{\epsilon}_b + B_g\tilde{G}_b \quad [7d]$$

where the B's are determined from the combustion zone theory. The particular case of the theory of Ref. 1 is worked out in detail in the Appendix. Solution of the boundary equations is moderately tedious, but one finally obtains the two expressions for the admittances Y and Y':

$$Y_b = -\frac{\bar{v}}{\bar{P}}\left\{\frac{\tilde{\mu}_b}{\tilde{\epsilon}_b}-\left[\frac{1}{\gamma}-\frac{\left[\frac{\tilde{\psi}_b}{\tilde{\epsilon}_b}-\left(\frac{\gamma-1}{\gamma}\right)\right]\left[1-\frac{\gamma M^2\Delta^2}{(1+\Delta)}\left(\frac{\tilde{\mu}_b}{\tilde{\epsilon}_b}\right)\right]}{\left[(1+\Delta)-\frac{\gamma M^2\Delta^2(1-\tilde{\psi}_b/\tilde{\epsilon}_b)}{(1+\Delta)}\right]}\right]\right\} \quad [8a]$$

and

$$Y_b' = -(\bar{v}/\bar{P})\,\{(\tilde{\mu}_b/\tilde{\epsilon}_b) - 1 + (\tilde{\psi}_b/\tilde{\epsilon}_b)\} \quad [8b]$$

Eq. [8b] can, of course, be obtained directly from Eq. [1c] by using the gas law, and this result serves mainly to give some confidence in the correctness of the lengthy algebraic manipulations. One can recognize the two special cases: 1) the isentropic situation for which $\tilde{\psi}_b/\tilde{\epsilon}_b \to (\gamma-1)/\gamma$ and $Y_b \to Y_b' \to -(\bar{v}/\bar{P})[(\tilde{\mu}_b/\tilde{\epsilon}_b) - (1/\gamma)]$, which is the result obtainable directly from the simple acoustic relationship between density and pressure; and 2) the low frequency limit for which $\tilde{\psi} \to 0$, $\Delta \to 0$, for which

$$Y_b \to -(\bar{v}/\bar{P})[(\tilde{\mu}_b/\tilde{\epsilon}_b) - 1]$$

which is the expected result for this limit.

An incidental result is also of some interest, however. One also may determine the density amplitude of the "entropy" wave. Solving the boundary equation for $\tilde{\sigma}_3$, one finds

$$\frac{\tilde{\sigma}_3}{\tilde{\epsilon}_2} = \frac{2(1+\Delta)^2\left[\frac{\gamma-1}{\gamma}-\frac{\tilde{\psi}_b}{\tilde{\epsilon}_b}\right]}{\gamma[1+\Delta(1+M)]\left\{\frac{1+\Delta}{\gamma}+M\left[1-\frac{\tilde{\psi}_b}{\tilde{\epsilon}_b}-\frac{\tilde{\mu}_b}{\tilde{\epsilon}_b}\right]-M\Delta\frac{\tilde{\mu}_b}{\tilde{\epsilon}_b}-M^2\Delta\left[1-\frac{\tilde{\psi}_b}{\tilde{\epsilon}_b}-\frac{\tilde{\mu}_b}{\tilde{\epsilon}_b}\right]\right\}} \quad [8c]$$

Note that $\tilde{\sigma}_3$ vanishes in the isentropic limit $\tilde{\psi}_b/\tilde{\epsilon}_b \to (\gamma-1)/\gamma$, as it should.

To complete the solution, it is necessary to express the temperature perturbation $\tilde{\psi}$ in terms of known quantities. Eq. [7d] provides one relationship between $\tilde{\psi}$, $\tilde{G}$, and $\tilde{\epsilon}$ at the boundary, and the condition that $\tilde{\epsilon}_4$ vanish provides the second such relationship, which then permits determination of $\tilde{\psi}_b$. Solving Eqs. [7a–7c] for the condition that $\tilde{\epsilon}_4 = 0$, one obtains

$$\frac{\tilde{\psi}_b}{\tilde{\epsilon}_b}-\frac{\gamma-1}{\gamma} = \frac{\left[\frac{-cM\tilde{G}_b}{i\omega\tilde{\epsilon}_b}-\frac{M^2(\gamma-1)}{1-M^2}\left(\frac{\tilde{\mu}_b}{\tilde{\epsilon}_b}-\frac{2}{\gamma}\right)\right]\left[1+\Delta-\frac{M^2\Delta^2}{1+\Delta}\right]}{\left[(1+\Delta)^2-M^2\left(\gamma\Delta^2-\frac{(\gamma-1)}{1-M^2}\right)-\frac{2M^4(\gamma-1)\Delta^2}{(1-M^2)(1+\Delta)}\right]} \quad [8d]$$

As one limiting case whose result is familiar, one may consider a rigid, nonpyrolyzing surface. In that limit, the behavior must be isentropic if the boundary heat loss is zero, i.e., if $\tilde{G}_b = 0$. Some care must be exercised in passing to the limit, but the correct result $(\psi/\epsilon) \rightarrow (\gamma - 1)/\gamma$ is finally obtained. In the slightly more general limiting case of a nonrigid, nonpyrolyzing surface (the ordinary acoustic problem), a small temperature gradient at the surface, x_b, will exist even in the isentropic case. That temperature gradient corresponds to the proper value for a sound wave. For the more general case $M \neq 0$, it can also be verified directly that the term

$$\left[\frac{cM}{i\omega}\frac{\tilde{G}_b}{\bar{\epsilon}_b} + \frac{M^2(\gamma - 1)}{1 - M^2}\left(\frac{\tilde{\mu}_b}{\bar{\epsilon}_b} - \frac{2}{\gamma}\right)\right]$$

vanishes for an isentropic field. Now by referring back to Eq. [7d], one may eliminate $\tilde{G}_b$, and the admittances are then expressed in terms of the quantities $\tilde{\mu}/\bar{\epsilon}$, B_p, and B_g, which must be obtained from theory of the combustion zone. There would seem to be little point actually in displaying the final result, in view of the complexity of the result. Substitution of Eq. [8d] into Eqs. [8a] and [8b] yields the final expressions for the admittances Y and Y'. In the following section, these results will be discussed in terms of the combustion zone theory of Ref. 1.

Discussion

Now that the expressions Y and Y' have been obtained, one turns toward a discussion of their use. The virtual acoustic admittance Y is the ratio of acoustic velocity to acoustic pressure at the burning surface. Thus, it specifies the boundary condition for sound (isentropic) waves at the propellant boundary and would be determined from experimental measurements carried out in the far zone where the anisentropic field vanishes. The quantity Y' specifies the ratio of fluctuating velocity to fluctuating pressure, including both isentropic and anisentropic components adjacent to the propellant surface (i.e., just outside the very thin induction gas phase reaction zone). Thus, Y' would be determined by direct measurements (of, say, the fluctuating pressure and velocity) in the near zone, close to the burning propellant. Eqs. [8a] and [8b] display the structure of these two admittances as determined by solution of the equation of fluid flow, in the linear approximation that is valid for small fluctuations from the steady state.

Since the form of Y' is the simpler, the discussion will begin by an examination of it. Calculation of the actual fluctuating temperature amplitude at x_b requires a theoretical treatment of time-dependent burning in the combustion sublayer. The most complete treatment yet available is that of Ref. 2, and in an attempt to evaluate the type of behavior which might be expected, Eq. [8b] has been calculated for several test cases using that theory. Typical results are shown in Figs. 2 and 3. The amplitude of the temperature response, as indicated by $\tilde{\psi}_b/\bar{\epsilon}$, is shown by the lowest curve of Fig. 2. The dashed curve displays the large brackets of Eq. [8a]. It is this bracket that has been set equal to $1/\gamma$ in previous treatments, and amplification at the propellant surface occurs when $Re(\tilde{\mu}_b/\bar{\epsilon}_b)$ lies above the dashed curve. It is clear that the boundary condition at x_b is not, in general, an isothermal one, even at moderately low frequencies of a few hundred cycles per second.

At first glance, this might seem rather startling because of the rather effective thermostating action provided by combustion. On second thought, however, one recognizes that the boundary conditions for the fluid dynamical equations include not only the temperatures but also the temperature gradients at the boundaries. Thus, it should not be surprising to observe $\tilde{\psi}_b/\bar{\epsilon}$ departing from zero at the relatively low frequencies for which time-dependent heat conduction in the solid phase becomes relevant. These considerations can be displayed analytically in a convenient way by confining one's attention to the usual regime of flow velocities and frequencies. Then it is possible to take advantage of the smallness of the quantities M^2 and Δ to obtain a relatively simple expression for the temperature response at x_b. Eqs. [8d], [7d], [A8], and [A9] combine to yield

$$\frac{\tilde{\psi}_b}{\bar{\epsilon}_b} \approx \frac{\left(\frac{\gamma - 1}{\gamma}\right) i\omega\tau_0(1 + \Delta) + \left[-\frac{J(\omega)\tilde{\mu}_0}{h\bar{\epsilon}_b} + \frac{\tilde{\mu}_1 - \tilde{\mu}_0}{\bar{\epsilon}_b} - \frac{i\omega\tau_0}{\gamma}\ln\left(\frac{c_pT_1 + h}{c_pT_0 + h}\right)\right]}{\frac{c_pT_f}{h} + i\omega\tau_0(1 + \Delta)}$$

It is the quantity $J(\omega)$ that contributes toward the significant magnitude of $\tilde{\psi}_b/\bar{\epsilon}$ at moderately low frequencies, and one recalls that $J(\omega)$ arises through consideration of time-dependent heat conduction in the solid. It becomes clear, therefore, that, in general, the temperature boundary condition at the downstream side of the combustion boundary layer corresponds to an isothermal one only at quite low frequencies. On the other hand, for frequencies such that the isothermal limit becomes invalid, the boundary condition is not a simple one, and the interpretation of a measurement of Y' in terms of either the mass response or the temperature response will require additional information such as, for example, the measurement of the acoustic admittance Y. It is clear from Eqs. [8a] and [8b] that determination of both Y and Y' permits determination of both $\tilde{\mu}_b/\bar{\epsilon}_b$ and $\tilde{\psi}_b/\bar{\epsilon}_b$.

From the point of view of the ability of the burning surface to amplify sound, the acoustic admittance Y is a quantity of central interest. If the real part of Y should be negative, then sound is amplified upon being reflected from the burning propellant surface, and instability will result unless acoustic losses elsewhere are sufficiently large. Unfortunately, perhaps, the structure of Y as displayed by Eq. [8a] is not terribly illuminating. One will recognize, however, that in the isentropic limit

$$\tilde{\psi}_b/\bar{\epsilon}_b \rightarrow (\gamma - 1)/\gamma$$

so that

$$Y_b \rightarrow -(\bar{v}/\bar{P})[(\tilde{\mu}_b/\bar{\epsilon}_b) - (1/\gamma)]$$

As has been noted, it has been conventional, in earlier discussions on this subject, to neglect the acoustic boundary layer in the theoretical calculation of Y. This is equivalent to assuming the isentropic limit, i.e., to ignoring the correction to $1/\gamma$ (in Eq. [8a]) which arises because of nonisentropic behavior of the gas near the boundary. Thus, it has been said that the propellant amplifies sound if $\tilde{\mu}_b/\bar{\epsilon}_b > 1/\gamma$. In order to explore the significance to acoustic stability of the nonisentropic acoustic boundary layer, the combustion zone theory of Ref. 2 has been used to calculate the quantities appearing in Eq. [8a]. Results for two such calculations are shown in Figs. 4 and 5. In each of these figures, the real part of y_b is shown by the solid line, and the real part of y_b' is shown by the dashed line. (These bracketed terms in Eq. [8a] would have been replaced by $1/\gamma$ in earlier treatments where the acoustic boundary layer was not considered, and the reduced admittance that would then have been obtained is shown by the dash-dot line.) Amplification would result whenever the solid curve lies above the zero axis, and an appreciable effect on the domain of amplification can occur, as illustrated by Fig. 5.

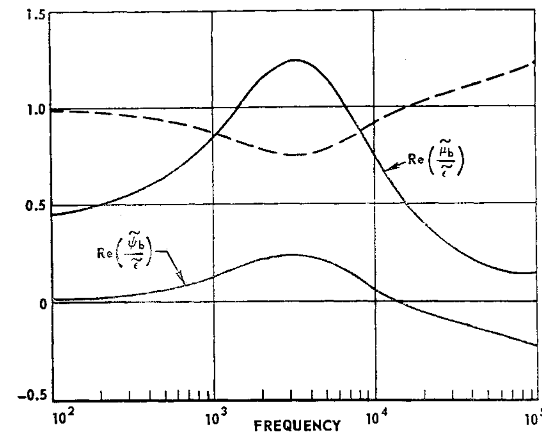

Fig. 2 Sample mass response and temperature response functions according to the theory of Ref. 2. In this example, the propellant parameters are $\bar{m}$ = 2.37 g/cm²-sec, j = 0.4, n = 0.43, $\bar{P}$ = 34 atm, T_0 = 1125°K, α = −0.5, T_c = 300°K, A_s = 40,000 cal/mole, ρ_s = 1.58 g/cm³, molecular weight = 24.8, T_1 = 2130°K, T_f = 2506°K, $\lambda = \lambda_s = 5 \times 10^{-4}$ cal/cm-sec-°K, γ = 1.23, h_v = 200 cal/g, $c_s = c_p = \frac{1}{3}$

The very low frequency behavior is, of course, expected. The quantity $\tilde{\mu}_b/\tilde{\epsilon}_b \rightarrow n$, the usual pressure index in the steady-state burning law, and $\tilde{\psi}_b/\tilde{\epsilon}_b \rightarrow 0$ as the steady state (zero frequency) is approached. Eq. [8a] can be simplified somewhat if attention is restricted to the usual burning velocity and frequency domain. Then it will be entirely reasonable to neglect terms of order $M^2\Delta$ compared with unity, and Eqs. [8a], [8d], and [7d] may be combined to obtain

$$Y \approx -(\bar{v}/\bar{P})[(\tilde{\mu}_b/\tilde{\epsilon}_b) - (1 - B_p)]$$

In the theory of Ref. 2, the quantity B_p, which measures the temperature response to fluctuating pressure, acquires a rather large negative real part as the frequency becomes very high. It is for this reason that the dashed lines of Figs. 2 and 3 show their monotone increase at high frequency, and such behavior probably should not be found if one used a combustion zone theory completely valid at very high frequencies.

Conclusion

In the light of such theory as now exists, it would seem that the domain of instability is somewhat compressed by the inclusion of nonisentropic behavior of the acoustic boundary sublayer. In so far as the function of theory is to interpret experimental measurement, it has been seen that the necessary structure exists, although that part which pertains to the combustion zone itself is undeniably very crude. Thus, measurement of the temperature response ($\tilde{\psi}_b/\tilde{\epsilon}$) as well as one of the admittances Y or Y' could be desirable, particularly in the high frequency domain, because it would permit comparison with mass response measurements without appeal to combustion zone theory.

Although measurement of acoustic admittance is entirely adequate for the question of propellant acoustic amplification, the theoretician and, hopefully, the experimenter will feel compelled to inquire as to whether or not the structure of the admittance as displayed by the theory is a sufficiently complete representation of the problem. With good fortune, features of the theory will be in agreement with experiment, and one may hope that the nature of the discrepancies will help to focus attention on the most important shortcomings in attempts to analyze and understand the results of an impressively complicated interplay between a variety of physicochemical processes in the combustion instability of solid fuel motors.

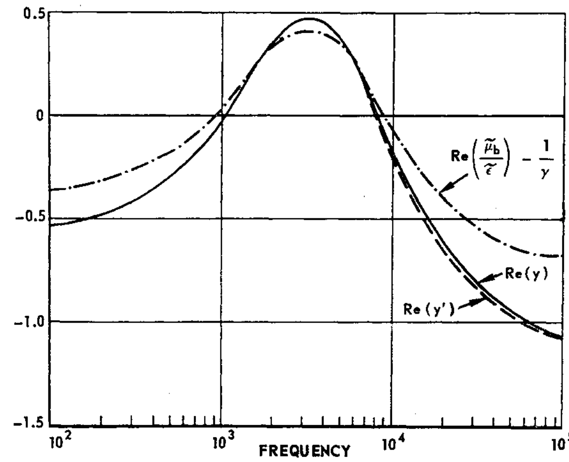

Fig. 4 Sample calculation showing the effect of the acoustic boundary layer on admittance for the propellant of Fig. 2. The dash-dot curve would have been obtained if the acoustic boundary layer had been ignored. Amplification occurs where the solid line lies above the zero axis

Acknowledgments

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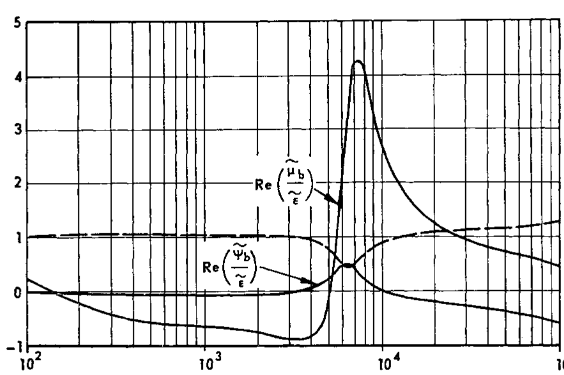

Fig. 3 Sample mass response and temperature response functions according to the theory of Ref. 2. In this example, the propellant parameters are the same as those of Fig. 2 except for the following: $\bar{m}$ = 1.52 g/cm²-sec, j = 2.0, n = 0.68, T_0 = 775°K, α = 1.0, ρ_s = 1.62 g/cm³, molecular weight = 26.8, T_1 = 2560°K, T_f = 3012, γ = 1.22

Appendix: Combustion Zone Relationship

In Refs. 1 and 2, the ignition temperature T_1 is assigned a fixed value, and expressions are derived for the mass flow rate

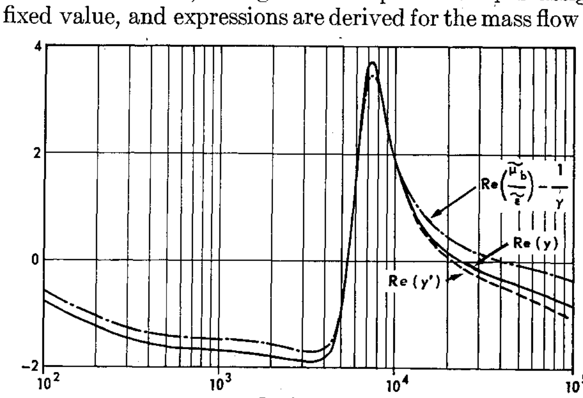

Fig. 5 Sample calculation similar to Fig. 4, except that the propellant corresponds to that of Fig. 3

at both x_1 and x_0. If these expressions and the conservation equations are used, boundary relationships at $x = x_f$ will be obtained as follows. Since the gas phase reactions themselves are assumed to be fast, $m_b = m_1$, i.e., $\mu_b = \mu_1$. The energy conservation equation across the thin zone $x_1 \rightarrow x_b$ yields the expression

$$[(\lambda\partial T/\partial x) - mC_pT]_{x_b} = [(\lambda\partial T/\partial x) - mC_pT - mQ]_{x_1} \quad \text{[A1]}$$

where Q is the heat released per unit mass in this zone by burning. Taking increments from the steady state and using the steady-state relationships

$$\overline{(\partial T/\partial x)_{x_b}} = \partial\bar{T}_f/\partial x = 0 \quad \text{[A2]}$$

$$-\bar{m}C_p\bar{T} = \lambda\overline{(\partial T/\partial x)}_{x_1} - \bar{m}C_pT_1 - \bar{m}Q \quad \text{[A3]}$$

and (cf., Ref. 1)

$$\lambda\overline{(\partial T/\partial x)}_{x_1} = \bar{m}(C_pT_1 + h) \quad \text{[A4]}$$

one finally obtains

$$\psi_b = \frac{\lambda}{\bar{m}C_p} G_b - \frac{\lambda T_1}{\bar{m}C_p\bar{T}_f} G_1 + \mu_1 \left[\frac{(C_pT_1 + h)}{C_p\bar{T}_f}\right] \quad \text{[A5]}$$

where $G_1 \equiv (\partial\psi/\partial x)T_1$. The quantity G_1 can be calculated from Ref. 1 as follows. Application of the equation of energy conservation across the induction zone gives directly

$$\widetilde{[\lambda(\partial T/\partial x)_{x=x_1} - mC_pT_1] - [\lambda(\partial T/\partial x)_{x=x_0} - m_0C_pT_0]} = i\omega C_v\bar{\rho}\bar{T}\tilde{\epsilon}\bar{x}_1 \quad \text{[A6]}$$

since the pressure is assumed constant across the thin induction zone, the mean thickness of which is $\bar{x}_1$. Using Eq. [6a] of Ref. 1 for $(\partial T/\partial x)_{x_0}$, perturbing Eq. [A6], and then using Eq. [A5] leads to the result

$$\tilde{\psi}_b = \frac{\lambda\tilde{G}_b}{\bar{m}C_p} + \tilde{\mu}_1 \left[\frac{C_pT_1 + h}{C_p\bar{T}_f}\right] - \left\{\frac{T_1}{\bar{T}_f}\tilde{\mu}_1 + \left[\frac{h + J}{C_p\bar{T}_f}\right]\tilde{\mu}_0 + \frac{i\omega\tau_0 h}{C_p\gamma\bar{T}_f}\tilde{\epsilon}\ln\left(\frac{C_pT_1 + h}{C_p\bar{T}_0 + h}\right)\right\} \quad \text{[A7]}$$

where $\tilde{\mu}_0$, $\tilde{\mu}_1$, and J are frequency-dependent quantities determined in Ref. 1. Eq. [A7] is the relationship between the temperature fluctuation and the temperature gradient fluctuation which is obtained from Ref. 1. Both $\tilde{\mu}_1$ and $\tilde{\mu}_0$ quantities are determined in terms of the pressure, so that one has, referring to Eq. [7c]

$$B_p = \frac{\tilde{\mu}_1}{\tilde{\epsilon}}\left[\frac{h}{C_p\bar{T}_f}\right] - \left[\frac{h + J}{C_p\bar{T}_f}\right]\left(\frac{\tilde{\mu}_0}{\tilde{\epsilon}}\right) - \frac{i\omega\tau_0 h}{C_p\gamma\bar{T}_f}\ln\left(\frac{C_pT_1 + h}{C_p\bar{T}_0 + h}\right) \quad \text{[A8]}$$

and

$$B_g = \lambda/\bar{m}C_p \quad \text{[A9]}$$

Analogous expressions are obtained if one uses Ref. 7, wherein $\tilde{\mu}_1$, $\tilde{\mu}_0$, and J must be replaced by J^+ and the values of $\tilde{\mu}_1$ and $\tilde{\mu}_0$ as calculated in Ref. 7.

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